

# ON ABELIAN 2-CATEGORIES AND DERIVED 2-FUNCTORS

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*Dedicated to the memory of Prof. V. K. Bentkus*

## 1. INTRODUCTION

We will assume that the reader is familiar with the work of Mathieu Dupont on abelian 2-categories [14] (called 2-abelian **Gpd**-category in [14]). We do not recall this quite technical definition and give only general remarks and few examples.

The main difference between abelian categories and abelian 2-categories is that in abelian 2-categories we have not only objects and morphisms but we have also 2-cells or tracks between two parallel arrows. To be more precise abelian 2-categories are first of all groupoid enrich categories.

Recall that a *groupoid* is a small category such that all morphisms are isomorphisms. For a groupoid  $\mathbb{G}$  and an object  $x \in \mathbb{G}$  we let  $\pi_0(\mathbb{G})$  and  $\pi_1(\mathbb{G}, x)$  be the set of connected components of  $\mathbb{G}$  and the group of automorphisms of  $x$  in  $\mathbb{G}$  respectively.

The prototype of abelian categories is the category of abelian groups **Ab**. A similar role in the two dimensional world is played by the 2-category **SCG** of symmetric categorical groups. The notion of a symmetric categorical group is a categorification of a notion of abelian group. More precisely, let  $(\mathbb{A}, +, 0, a, l, r, c)$  be a symmetric monoidal category, where  $+: \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$  is the composition law,  $0$  is the neutral element,  $a$  is the associative constraints,  $c$  is the commutativity constraints and  $l: \text{Id} \rightarrow 0 + \text{Id}$  and  $r: \text{Id} \rightarrow \text{Id} + 0$  are natural transformations satisfying well-known properties [23]. We will say that  $\mathbb{A}$  is a *symmetric categorical group* or *Picard category* provided  $\mathbb{A}$  is a groupoid and for any object  $x$  the endofunctor  $x+ : \mathbb{A} \rightarrow \mathbb{A}$  is an equivalence of categories. It follows that  $\pi_0(\mathbb{G})$  is an abelian group and the endofunctor  $x+ : \mathbb{A} \rightarrow \mathbb{A}$  yields an isomorphism  $\pi_1(\mathbb{G}, 0) \rightarrow \pi_1(\mathbb{G}, x)$  of abelian groups. In what follows we will write  $\pi_1(\mathbb{G})$  instead of  $\pi_1(\mathbb{G}, 0)$ .

Symmetric categorical groups form the 2-category **SCG** which is the prototype for abelian 2-categories. In an abelian 2-category  $\mathfrak{T}$  for any two object  $A$  and  $B$  the hom groupoid **Hom** $_{\mathfrak{T}}(A, B)$  is in fact a symmetric categorical group. Moreover  $\mathfrak{T}$  has kernels and cokernels in the 2-dimensional sense and satisfies exactness properties.

For any object  $A$  of an abelian 2-category  $\mathfrak{T}$ , the composition of morphisms equips the symmetric categorical group **Hom** $(A, A)$  with a multiplication. This two structure form a mathematical object called 2-ring. Here a 2-ring (called also Ann-category [28], or categorical ring [20]) is a categorification of the version of a ring and it consists of a symmetric categorical group equipped with "multiplication" satisfying Laplaza coherent

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axioms [22]. This notion with different name already presents in the pioneering work of Takeuchi [32].

Large class of examples of abelian 2-categories are given by the 2-categories of 2-modules over 2-rings [14], [30], [18].

Any category can be considered as a groupoid enrich category with trivial tracks. Hence the theory of groupoid enrich categories generalizes usual category theory. In a sense the theory of abelian 2-categories also generalizes the theory of abelian categories, but in a more tricky way. Let  $\mathbf{A}$  be an abelian category, we wish to associate to it an abelian 2-category. The first idea which comes in mind is to consider  $\mathbf{A}$  as a groupoid enrich category with trivial tracks. However in this way we never obtain an abelian 2-category except the trivial case  $\mathbf{A} = 0$  [14]. More interesting is the following construction. Let  $\mathbf{A}$  be an abelian category and consider the 2-category  $\mathbf{A}^{[1]}$  consisting of arrows  $A = (A_1 \xrightarrow{a} A_0)$  considered as chain complexes concentrated in dimensions 1 and 0. Then morphisms in  $\mathbf{A}^{[1]}$  are just chain maps and tracks are just chain homotopies. However  $\mathbf{A}^{[1]}$  is usually NOT an abelian 2-category, except the case when  $\mathbf{A}$  is semi-simple [14]. To solve the dilemma, let us assume that  $\mathbf{A}$  is an abelian category with enough projective objects and consider the full 2-subcategory  $\mathbf{A}_c^{[1]}$  consisting of arrows  $A = (A_1 \xrightarrow{a} A_0)$  with projective  $A_0$ . Then  $\mathbf{A}_c^{[1]}$  is an abelian 2-category [25] and this is a way how abelian categories should be considered as a part of the theory of abelian 2-categories. To support this point of view let us observe that any abelian 2-category  $\mathfrak{T}$  defines the derived category  $\mathbb{D}(\mathfrak{T})$ , which is triangulated category in the usual sense [35]. It is easy to observe that the derived category of  $\mathbf{A}_c^{[1]}$  triangulated equivalent to the classical derived category  $\mathbb{D}(\mathbf{A})$  of  $\mathbf{A}$ .

It is interesting to compare  $\mathbf{A}^{[1]}$  and  $\mathbf{A}_c^{[1]}$  from the point of view of usual category theory. The underlying category of  $\mathbf{A}^{[1]}$  is abelian, while the underling category of  $\mathbf{A}_c^{[1]}$  has no kernels nor cokernels in general. However from the point of view of 2-dimensional algebra  $\mathbf{A}_c^{[1]}$  is much nicer. This is no accident and the reader familiar with homological and homotopical algebra recognize the role of cofibrant objects.

Of course the dual construction works as well. Namely, if  $\mathbf{A}$  is an abelian category with enough injective objects then the 2-category  $\mathbf{A}_f^{[1]}$  consisting of arrows  $A = (A_1 \xrightarrow{a} A_0)$  with injective  $A_1$  is an abelian 2-category. Assuming now that  $\mathbf{A}$  has enough projective and injective objects. So we could perform both constructions  $\mathbf{A}_c^{[1]}$  and  $\mathbf{A}_f^{[1]}$  and they looks quite different from the point of view of the usual category theory but they are 2-equivalent abelian 2-categories (actually using "butterflies" [2] these constructions can be unify and get an abelian 2-category starting from an arbitrary abelian category satisfying some smallness conditions).

It is maybe worthwhile to say that the notion of 2-equivalence is a 2-dimensional analogue of the notion of equivalence of categories. The set of objects is NOT invariant under equivalence of categories. Similarly the underlain category of an abelian 2-category is NOT invariant under 2-equivalences as we have seen in the examples  $\mathbf{A}_c^{[1]}$  and  $\mathbf{A}_f^{[1]}$ . Now we give more examples of the same phenomena.

Assume  $R$  is a ring. The category  $\mathbf{A}$  of left  $R$ -modules is an abelian category, so we can consider the corresponding abelian 2-category  $\mathbf{A}_c^{[1]}$ . On the other hand  $R$  can be seen as a discrete 2-ring. So we have another abelian 2-category, namely the 2-category of 2-modules over  $R$ . These 2-categories are very different from the point of view of the usual category theory but they are 2-equivalent abelian 2-categories (see Corollary 2).

It is well-known and is absolutely trivial that the category of abelian groups is isomorphic to the category of modules over the ring of integers. Probably a similar fact is also true for symmetric categorical groups. The role of integers should be played by the free symmetric categorical group with one generator equipped with appropriate multiplication. After my suggestion Vincent Schmitt worked on this and related problems, but unfortunately his work [30] in this direction is unfinished.

However, if one considers the problem not up to isomorphisms but up to 2-equivalences, then it can be solved quite easily. Namely, one can consider the following symmetric categorical group  $\Phi$ . Objects of  $\Phi$  are integers. If  $n \neq m$  then there is no morphism from  $n$  to  $m$ ,  $n, m \in \mathbb{Z}$ . The group of automorphisms of  $n$  is the cyclic group of order two with generator  $\epsilon_n$ ,  $n \in \mathbb{Z}$ . The monoidal functor in  $\Phi$  is induced by the addition of integers. The associativity and unite constraints are identity morphisms, while the commutativity constraint  $n + m \rightarrow m + n$  equals to  $nm\epsilon_{n+m}$ . We will see that  $\Phi$  plays the same role in  $\mathfrak{SCG}$  as the abelian group of integers plays in the category of abelian groups in the following sense:  $\Phi$  has a natural 2-ring structure induced by the multiplication of integers (all distributivity constraints being trivial) and the 2-category of 2-modules over  $\Phi$  in fact is 2-equivalent to the 2-category  $\mathfrak{SCG}$  (see Proposition 13).

These facts can be easily deduced from two theorems proved below. The first one is a 2-dimensional analogue of the Gabriel-Mitchel theorem (see Theorem 1 below) while the second theorem claim that abelian 2-category  $\mathfrak{SCG}$  (as well as the abelian 2-category of 2-modules over a 2-ring) has enough projective objects. In fact it has also enough injective objects. The result on projective and injective objects first was proved in [26] and this works should be considered as an extended version of it. We also added small sections on resolutions and derived 2-functors, following to [6], [7] and [13]. In particular we develop the theory of secondary ext objects in abelian 2-categories and we show that the cohomology of 2-groups can be described via such ext.

## 2. GABRIEL-MITCHEL THEOREM

We start with some definitions. Following to [14] an *additive 2-functor* from an abelian 2-category  $\mathfrak{T}$  to another abelian 2-category  $\mathfrak{T}_1$  is a pseudo-functor which on hom-s is a morphism of symmetric monoidal categories. A 2-functor  $F : \mathfrak{T} \rightarrow \mathfrak{T}'$  is called a *2-equivalence* between abelian 2-categories if the functors  $\mathfrak{T}(A, A') \rightarrow \mathfrak{T}'(FA, FA')$  are equivalences of groupoids for all objects  $A, A'$  of  $\mathfrak{T}$  and each object  $B$  of  $\mathfrak{T}'$  is equivalent to some object of the form  $F(A)$ . If  $\mathfrak{T}$  is an abelian 2-category, we let  $\mathbf{Ho}(\mathfrak{T})$  be the category which has the same objects as  $\mathfrak{T}$ , while for objects  $A$  and  $B$  we have  $\text{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, B) = \pi_0 \mathbf{Hom}_{\mathfrak{T}}(A, B)$ . This category is known as the *homotopy category* of  $\mathfrak{T}$ . A morphism  $f : A \rightarrow B$  in an abelian 2-category  $\mathfrak{T}$  is called *faithful* provided for all object  $X$  the induced functor

$f^X : \mathfrak{T}(X, A) \rightarrow \mathfrak{T}(X, B)$  is faithful. An object  $A$  in an abelian 2-category  $\mathfrak{T}$  is called *injective* provided for any faithful morphism  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  the induced map

$$\mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(S_2, A) \rightarrow \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(S_1, A)$$

is surjective. We will say that an abelian 2-category  $\mathfrak{T}$  has enough injective objects provided for any object  $\mathbb{A}$  there exist a faithful morphism  $\mathbb{A} \rightarrow \mathbb{S}$  with injective object  $\mathbb{S}$ .

Dually, a morphism  $f : A \rightarrow B$  in an abelian 2-category  $\mathfrak{T}$  is called *cofaithful* provided for all object  $X$  the induced functor  $f_X : \mathfrak{T}(A, X) \rightarrow \mathfrak{T}(B, X)$  is faithful. An object  $A$  is called *projective* provided for any cofaithful morphism  $F : S_1 \rightarrow S_2$  the induced map

$$\mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, S_1) \rightarrow \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, S_2)$$

is surjective. We will say that an abelian 2-category  $\mathfrak{T}$  has enough projective objects provided for any object  $A$  there exist a cofaithful morphism  $S \rightarrow A$  with projective object  $S$ .

A *coproduct* of the family  $A_i, i \in I$  of objects in an abelian 2-category  $\mathfrak{T}$  is an object  $A = \bigoplus_{i \in I} A_i$  equipped with maps  $\mu_i : A_i \rightarrow A, i \in I$  such that for any object  $X \in \mathfrak{T}$  the induced morphism

$$\mu^* : \mathbf{Hom}_{\mathfrak{T}}(A, X) \rightarrow \prod_{i \in I} \mathbf{Hom}_{\mathfrak{T}}(A_i, X)$$

is an equivalence of groupoids. In this case  $A$  is coproduct of the family  $A_i, i \in I$  in the category  $\mathbf{Ho}(\mathfrak{T})$  as well. By duality we can also talk on products. Observe that if  $A_1$  and  $A_2$  are objects in an abelian 2-category  $\mathfrak{T}$  then there exist a product  $A = A_1 \times A_2$  which is also a coproduct. We will say that an abelian 2-category  $\mathfrak{T}$  has coproducts if for any family of objects  $A_i, i \in I$  there exists coproduct  $\bigoplus_{i \in I} A_i$ . Dually for products. It is obvious that coproduct of projective objects is projective and product of injective objects is injective.

It is easy to show that the abelian 2-category  $\mathfrak{SCG}$  has all coproducts and products. Similarly for the 2-category of 2-modules over 2-rings.

We will say that an object  $G$  is a *generator* of an abelian 2-category  $\mathfrak{T}$  provided for any object  $S$  there is a diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \uparrow \alpha & \searrow & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

such that the relative cokernel of this diagram is equivalent to  $S$  and objects  $A, B, C$  are coproducts of  $G$ . An object  $S$  is *small* if  $\mathbf{Hom}_{\mathfrak{T}}(\mathbb{S}, -)$  preserves coproducts. The reader familiar with the corresponding notion in the classical world maybe find this definition ad hoc. However, it is known (see for example pp. 52-53 in [3]) that an object  $G$  of an abelian category  $\mathbb{A}$  is a generator if and only if any object of  $\mathbb{A}$  is isomorphic to the cokernel of a map  $G_1 \rightarrow G_0$ , where  $G_1, G_2$  are coproducts of  $G$ .

In an abelian 2-category  $\mathfrak{T}$  any morphism has a kernel and cokernel in the sense of 2-categories. The cokernel of the morphism  $X \rightarrow 0$  is denoted by  $\Sigma X$ , while the kernel of the morphism  $0 \rightarrow X$  by  $\Omega X$ . It is well-known that  $\Omega^2 = 0$  and  $\Sigma^2 = 0$ . Objects  $X$  is called

*discrete* (resp. *codiscrete*, or *connected*) if  $\Omega X = 0$  (resp.  $\Sigma X = 0$ ). The full subcategory of  $\mathbf{Ho}(\mathfrak{T})$  formed by discrete (resp. codiscrete) objects is an abelian category denoted by  $\mathbf{Dis}(\mathfrak{T})$  (resp.  $\mathbf{Codis}(\mathfrak{T})$ ). It is a remarkable fact that  $\mathbf{Dis}(\mathfrak{T})$  and  $\mathbf{Codis}(\mathfrak{T})$  are equivalent categories.

The following is a 2-dimensional analogue of the classical Gabriel-Mitchel's theorem (p.54 in [3]). To state it recall that if  $\mathfrak{T}$  is an additive 2-category and  $M$  is an object in  $\mathfrak{T}$  then one has the 2-ring  $\mathbf{End}_{\mathfrak{T}}(M) := \mathbf{Hom}_{\mathfrak{T}}(M, M)$  (compare [14],[30]) and for any object  $X$ , the symmetric categorical group  $\mathbf{Hom}(M, X)$  is a right 2-module over  $h(X) = \mathbf{Hom}_{\mathfrak{T}}(M, X)$ . In this way we get a 2-functor  $h$  from  $\mathfrak{T}$  to the 2-category of 2-modules over  $\mathbf{End}_{\mathfrak{T}}(M)$ .

**Theorem 1.** *Let  $\mathfrak{T}$  be an abelian 2-category with arbitrary coproducts. If  $M$  is a small projective generator in  $\mathfrak{T}$  then 2-functor  $h$  from  $\mathfrak{T}$  to the category of right categorical modules over the categorical ring  $R = \mathbf{End}(M)$  is a 2-equivalence of abelian 2-categories.*

*Proof.* Our argument is almost literary follows to pp. 54-55 in [3]. The fact that if  $R$  is a 2-ring, then  $R$  considered as a right  $R$ -module is a small projective generator is an easy consequence of the 2-dimensional Yoneda lemma. Assume now that  $\mathfrak{T}$  satisfies the conditions of the theorem. We need to establish two facts. Firstly, for any objects  $X, Y$  in  $\mathfrak{T}$  the functor

$$h_{X,Y} : \mathbf{Hom}_{\mathfrak{T}}(X, Y) \rightarrow \mathbf{Hom}_R(hX, hY)$$

is an equivalence of categories and secondly every 2-module  $Z$  is equivalent to some  $h(X)$ .

We start to verify the first assertion. We fix  $Y$  and consider  $h_{X,Y}$  as a natural transformation  $\alpha : T \rightarrow S$ , where  $T(X) = \mathbf{Hom}_{\mathfrak{T}}(X, Y)$  and  $S(X) = \mathbf{Hom}_R(hX, hY)$ . Observe that if  $X = M$  then  $\alpha(X)$  is an equivalence of categories by Yoneda and because of smallness assumption it is also an equivalence for  $X$  isomorphic to a coproduct of  $M$ . Since hom-s are left exact and  $\mathbf{Hom}_{\mathfrak{T}}(M, -)$  is an exact functor (due to projectivity of  $M$ ) it follows that both 2-functors  $T$  and  $S$  are left exact as well. Since  $M$  is a generator it follows that  $\alpha(X)$  is an equivalence of categories for any  $X$ .

To see the second assertion, we use the fact that  $R$  is a generator in the 2-category of  $R$ -modules, hence we can write  $Z$  as a relative cokernel of a diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \alpha & \swarrow & \\ R^{(I_1)} & \xrightarrow{f} & R^{(I_2)} & \xrightarrow{g} & R^{(I_3)} \end{array}$$

where  $R^{(I)}$  denotes the coproduct of  $I$ -copies of  $R$ . Since

$$\mathbf{Hom}_R((hM)^{(I_i)}, ((hM)^{(I_j)})) \cong \mathbf{Hom}_R((h(M^{(I_i)}), (h(M^{(I_j)})))$$

we see that the diagram is equivalent to one which comes from a similar diagram in  $\mathfrak{T}$  by applying the 2-functor  $h$  and again by exactness of  $h$  we see that  $Z$  is equivalent to one of the form  $h(X)$ .  $\square$

As an immediate consequence of this theorem we see that in terminology [14] any 2-abelian  $\mathbf{Gpd}$ -category which posses a small projective generator and arbitrary coproducts

is automatically good 2-abelian **Gpd**-category. Another important consequence is the following.

**Corollary 2.** *Let  $R$  be a classical ring and  $\mathbb{A}$  be the abelian category of classical  $R$ -modules. Then the abelian 2-category  $\mathbb{A}_c^{[1]}$  is equivalent to the 2-category of 2-modules over  $R$  considered as a discrete 2-ring.*

*Proof.* Consider the following object  $R[0] = (0 \rightarrow R)$  of the abelian 2-category  $\mathbb{A}_c^{[1]}$ . It is obvious that  $R[0]$  is a small and projective in the abelian 2-category  $\mathbb{A}_c^{[1]}$ , to show that it is also a generator take any object  $A = (A_1 \rightarrow A_0)$  in  $\mathbb{A}_c^{[1]}$  and consider a projective resolution  $P_* \rightarrow A_1$  in the classical sense. Based on the description of cokernels of morphism of  $\mathbb{A}_c^{[1]}$  given in [25] it is obvious that  $A$  is equivalent to the relative cokernel of the diagram

$$\begin{array}{ccccc} & & 0 & & \\ & \searrow & \uparrow \alpha & \swarrow & \\ P_1[0] & \longrightarrow & P_0[0] & \longrightarrow & A_0[0] \end{array}$$

where  $\alpha$  is the trivial track. So we can apply Theorem 1 and the fact that the 2-ring  $\mathbf{Hom}_{\mathbb{A}_c^{[1]}}(P[0], P[0])$  is isomorphic to  $R$  considered as a discrete 2-ring to finish the proof.  $\square$

### 3. MORITA EQUIVALENCE

Two 2-rings  $\mathbb{R}$  and  $\mathbb{S}$  are called *Morita equivalent* provided the 2-categories of 2-modules over  $\mathbb{R}$  and  $\mathbb{S}$  are 2-equivalent. Based on the tensor product developed by Vincent Schmitt [30] lots of materials on Morita contexts [3] have 2-dimensional analogues. Details left to the interesting readers. Here we restrict ourself only with the following consequence of our version of the Gabriel-Mitchel theorem.

**Theorem 3.**  *$\mathbb{R}$  and  $\mathbb{S}$  are Morita equivalent iff there exist a small projective generator  $P$  in the 2-category of 2-modules over  $\mathbb{R}$  such that two rings  $\mathbb{S}$  and  $\mathbf{Hom}_{\mathbb{R}}(P, P)$  are equivalent.*

*Proof.* It suffices to observe that under 2-equivalences small projective generators corresponds to small projective generators and then use Theorem 1.  $\square$

### 4. 2-CHAIN COMPLEXES

In this section we fix an abelian 2-category  $\mathfrak{T}$ . In particular for morphisms  $f, g : A \rightarrow B$  there are defined morphisms  $f + g : A \rightarrow B$  and  $-f : A \rightarrow B$ .

A 2-chain complex  $(A_*, d, \partial)$  in  $\mathfrak{T}$  is a diagram of the form

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \curvearrowright & & \curvearrowright & & \\
 & & \partial_n \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & A_{n+2} & \xrightarrow{d_{n+1}} & A_{n+1} & \xrightarrow{d_n} & A_n & \xrightarrow{d_{n-1}} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \partial_{n-1} & & & & & & \\
 & & 0 & & 0 & & & & & & 
 \end{array}$$

i. e., a sequence of objects  $A_n$ , maps  $d_n : A_{n+1} \rightarrow A_n$  and tracks  $\partial_n : d_n d_{n+1} \Rightarrow 0$ ,  $n \in \mathbb{Z}$ , such that for each  $n$  the tracks

$$d_{n-1} d_n d_{n+1} \xRightarrow{d_{n-1} \partial_n} d_{n-1} 0 \xRightarrow{\quad} 0$$

and

$$d_{n-1} d_n d_{n+1} \xRightarrow{\partial_{n-1} d_{n+1}} 0 d_{n+1} \xRightarrow{\quad} 0$$

coincide.

For any 2-chain complex  $(A_*, d, \partial)$  and any integer  $n$ , there is a well-defined object called  $n$ -th homology  $\mathbf{H}_n(A_*)$  of  $A_*$  (see p. 138 [14]). Following to [6] we call  $\mathbf{H}_*(A_*)$  the *secondary homology* of  $A_*$ . Let us recall the definition. Let  $\mathbf{Z}_n(A_*)$  be the relative kernel  $\text{Ker}(d_n, \partial_{n-1})$ . Then we get a natural morphism  $k : A_{n+1} \rightarrow \mathbf{Z}_n$  and by the definition the secondary homology  $\mathbf{H}_n$  is the relative cokernel of the diagram

$$\begin{array}{ccc}
 & & 0 \\
 & \curvearrowright & \\
 \mathbf{A}_{n+2} & \xrightarrow{d_{n+1}} & A_{n+1} \xrightarrow{k} \mathbf{Z}_n \\
 & \alpha \uparrow & 
 \end{array}$$

In the abelian 2-category  $\mathfrak{SCS}$  we put

$$H^n(A_*) := \pi_0(\mathbf{H}_n(A_*))$$

These groups are known as *Takeuchi-Ulbrich homology* [13].

One of the main properties of the secondary homology is that it associates to an extension of 2-chain complexes a long 2-exact sequence of secondary homologies [13]. This implies also usual exactness of the long exact sequence of Takeuchi-Ulbrich homologies. The following easy lemma on 2-exact sequences will be useful in the section on derived 2-functors.

**Lemma 4.** *If*

$$\cdots \rightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \xrightarrow{h_n} A_{n-1} \rightarrow \cdots$$

*is a 2-exact sequence of symmetric categorical groups, then*

$$\pi_0(A_n) \cong \pi_1(A_{n-1})$$

*If  $\pi_1 A_n = 0$  (resp.  $\pi_0 A_{n-1} = 0$ ) then  $B_n \cong \text{Ker}(h_n)$  (resp.  $C \cong \text{Coker}(f_n)$ ).*

If  $A_*$  and  $B_*$  are 2-chain complexes in a 2-abelian category  $\mathfrak{T}$ , one can introduce a new 2-chain complex  $(\mathbf{Hom}(A_*, B_*), d, \partial)$  whose  $n$ -th component is the product

$$\mathbf{Hom}(A_*, B_*)_n = \prod_p \mathbf{Hom}_{\mathfrak{T}}(A_p, B_{n+p})$$

The boundary map

$$d : \mathbf{Hom}(A_*, B_*)_n \rightarrow \mathbf{Hom}(A_*, B_*)_{n-1}$$

is given by

$$(df)_p = df_p + (-1)^{p+1} f_{p-1}d$$

with following track

$$dd(f) \Rightarrow d^2f + fd^2 \Rightarrow 0$$

where the first track comes from the distributivity law, while the second is the sum  $f^*(\partial) + f_*(\partial)$ .

The objects of the symmetric categorical group  $\mathbf{H}_0(\mathbf{Hom}(A_*, B_*))$  are known as 2-chain maps from  $A_*$  to  $B_*$ , while morphisms of  $\mathbf{H}_0(\mathbf{Hom}(A_*, B_*))$  form tracks of 2-chain maps. We let  $\mathbf{2Chain}(\mathfrak{T})$  be the 2-category of 2-chain complexes, 2-chain maps and tracks of 2-chain maps. The corresponding homotopy category  $\mathbf{Ho}(\mathbf{2Chain}(\mathfrak{T}))$  is the homotopy category of 2-chain complexes. Thus a 2-chain map is a pair  $(f, \phi)$ , where  $f = (f_n)$  is a sequence of morphisms  $f_n : A_n \rightarrow B_n$  and  $\phi = (\phi_n)$  is the sequence of tracks  $f_n d_n \Rightarrow d_n f_{n+1}$  satisfying an obvious coherence condition, similarly tracks  $(f, \phi) \Rightarrow (f', \phi')$  are also pairs  $(h, \psi)$ , where  $h = (h_n)$  is the sequence of maps  $h_n : A_n \rightarrow B_{n+1}$  and  $\psi = (\psi_n)$  is the sequence of tracks  $d_n h_n + f_n \Rightarrow g + h_{n-1} d_{n-1}$  satisfying an obvious coherence condition.

The secondary homology  $\mathbf{H}_n$  defines an additive 2-functor

$$\mathbf{H}_n : \mathbf{2Chian}(\mathfrak{T}) \rightarrow \mathfrak{T}$$

In particular the Tackeuchi-Ulbrich homology is homotopy invariant, meaning that it factors through the homotopy category  $\mathbf{Ho}(\mathbf{2Chain}(\mathfrak{T}))$  of 2-chain complexes.

Observe that if  $T : \mathfrak{T} \rightarrow \mathfrak{T}'$  is an additive 2-functor between abelian 2-categories it induces a well-defined additive 2-functor  $\mathbf{2Chian}(\mathfrak{T}) \rightarrow \mathbf{2Chian}(\mathfrak{T}')$ , which is two-dimensional analogue on the degreewise action of an additive functor on chain complexes.

A 2-chain map  $(f_*, \phi_*) : A_* \rightarrow B_*$  is called *weak equivalence* provided the induced morphism in secondary homology  $\mathbf{H}_*(A_*) \rightarrow \mathbf{H}_*(B_*)$  is an equivalence. We let  $\mathbb{D}(\mathfrak{T})$  be the localization of the category  $\mathbf{Ho}(\mathbf{2Chain}(\mathfrak{T}))$  with respect to weak equivalences. Here we emphasise that the category  $\mathbb{D}(\mathfrak{T})$  exists provided  $\mathfrak{T}$  possesses countable coproducts and have enough projective objects. This can be proved essentially by repeating the argument of Samson Sanzidze given in [29]. It is not surprising that the category  $\mathbb{D}(\mathfrak{T})$  when it exists has a canonical triangulated category structure which is induced by the following mapping cone construction. Let  $(f_*, \phi_*) : (A_*, d^A, \partial^A) \rightarrow (B_*, d^B, \partial^B)$  be a 2-chain map. Define 2-chain complex  $(C_*, d, \partial)$  by

$$C_n = A_{n-1} \oplus B_n$$

$$d_n = \begin{pmatrix} -d^A & f \\ 0 & d^B \end{pmatrix}$$



while  $\partial d^2 \Rightarrow 0$  is the composite of the canonical track (coming from the distribution)  $d^2 \Rightarrow \begin{pmatrix} d^2 & -df + fd \\ 0 & d^2 \end{pmatrix}$  and by the track  $\begin{pmatrix} \partial^A & \phi \\ 0 & \partial^B \end{pmatrix}$ . The triangulated category  $\mathbb{D}(\mathfrak{T})$  is called the derived category of an abelian 2-category  $\mathfrak{T}$ .

We claim that this construction is not only an analogue of the classical construction of derived categories of abelian categories but also generalizes it. In fact, if  $\mathbf{A}$  is an abelian category with enough projective objects then the derived category  $\mathbb{D}(\mathbf{A})$  in the classical sense and the derived category  $\mathbb{D}(\mathbf{A}_c^{[1]})$  in the new sense are triangulated equivalent. The equivalence is given by the following functors. Define a triangulated functor  $\mathbb{D}(\mathbf{A}_c^{[1]}) \rightarrow \mathbb{D}(\mathbf{A})$  by the "total complex" construction (see for example Example 2. 11 in [6]). The functor in opposite direction is given as follows. If  $X_*$  is a chain complex in  $\mathbf{A}$ , first we have to replace it by a weak equivalent one which consists of projective objects (here one needs to assume that  $\mathbf{A}$  has countable coproducts) and then apply the functor  $P \mapsto P[0]$  to obtain a 2-chain complex. Here  $P[0] = (0 \rightarrow P)$  is an object in  $(\mathbf{A})_c^{[1]}$ . One sees that these constructions are mutually quasi-inverse functors.

It is a remarkable fact that the construction of a triangulated category from an abelian 2-category is intimately related to the derived category construction in the brave new algebra. Namely, if  $R$  is a 2-ring and  $H(R)$  is the corresponding ring-spectrum constructed in [20] then the derived category of 2-modules is triangulated equivalent to the derived category of  $H(R)$  in the brave new-algebra sense. The proof of this fact will appear in a forthcoming paper [27]. In particular, there is a ring spectrum  $\Lambda = H(\Phi)$ , which is characterized by the following properties:  $\pi_0(\Lambda) = \mathbb{Z}$ ,  $\pi_1(\Lambda) = \mathbb{Z}/2\mathbb{Z}$ ,  $\pi_i(\Lambda) = 0$ , if  $i \neq 0, 1$  and the first Postnikov invariant (as a ring spectrum) of  $\Lambda$  is the generator of the third MacLane (i.e. topological Hochschild) cohomology  $\mathbf{HML}^3(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ . Then  $\mathbb{D}(\mathfrak{SCS})$  and  $\mathbb{D}(\Lambda)$  are triangulated equivalent.

## 5. RESOLUTIONS AND DERIVED 2-FUNCTORS

Let  $\mathfrak{T}$  be an abelian 2-category with enough projective objects and  $A$  be an object in  $\mathfrak{T}$ . Following to [6] a *left  $A$ -augmented 2-chain complex* is a 2-chain map  $(\epsilon, \hat{\epsilon}) : A_* \rightarrow A$  of 2-chain complexes, where  $A$  is considered as a 2-chain complex concentrated in dimension 0 with trivial differentials  $d = 0, \partial = 0$  and  $(A_*, d, \partial)$  is a 2-chain complex with  $A_n = 0$ ,  $n < 0$ . Moreover  $\partial_n$  equal to the identity track for  $n < 0$  [6]. A left  $A$ -augmented 2-chain complex  $(\epsilon, \hat{\epsilon}) : A_* \rightarrow A$  is called *left resolution* provided  $(\epsilon, \hat{\epsilon})$  is a weak equivalence. A left resolution is called *projective resolution* provided  $A_n$  is projective for all  $n \geq 0$ . This notion is a particular case of a more general notion given in [6] and corresponds to the case when (in the notations [6])  $\mathbf{b}$  coincides with the class of all projective objects.

Then we can summarize some results of [6] in the following proposition.

**Proposition 5.** *Let  $\mathfrak{T}$  be an abelian 2-category with enough projective objects and let  $\mathbf{PR}$  be the full 2-subcategory of the 2-category  $\mathbf{2Chian}(\mathfrak{T})$  consisting of projective resolutions. Then the 2-functor given by taking the 0-th secondary homology defines a 2-equivalence*

$$\mathbf{PR} \rightarrow \mathfrak{T}$$

For the reader convenient we recall how to construct projective resolutions. Take an object  $A \in \mathfrak{T}$  and choose a projective object  $P_0$  together with a cofaithful morphism  $\epsilon : P_0 \rightarrow A_0$ . Let  $(K_0, i_0, \kappa_0)$  be the kernel of  $\epsilon$ , where  $i_0 : K \rightarrow P_0$  is a morphism and  $\kappa_0 : \epsilon \circ i_0 \Rightarrow 0$  is a track. Choose a projective object  $P_1$  together with a cofaithful morphism  $\epsilon_1 : P_1 \rightarrow K_0$ . Now we set  $d_0 = i_0 \epsilon_1 : P_1 \rightarrow P_0$  and  $\hat{\epsilon} = \epsilon_1^*(\kappa_0)$  which is a track  $\epsilon \circ d_0 \Rightarrow 0$ . Let  $(K_1, i_1, \kappa_1)$  be the relative kernel  $\text{Ker}(d_0; \hat{\epsilon})$ . Choose a projective object  $P_2$  together with a cofaithful morphism  $\epsilon_2 : P_2 \rightarrow K_1$  and set  $d_1 = i_1 \epsilon_2 : P_2 \rightarrow P_1$ ,  $\partial_0 = \epsilon_2^*(\kappa_1)$  and continue this way.

Let  $\mathbf{P}$  be the full 2-subcategory of  $\mathfrak{T}$  which consists of projective objects. Then for any additive 2-functor  $T : \mathbf{P} \rightarrow \mathfrak{SCEG}$  one obtains a well-defined additive 2-functors  $\mathbf{L}_n(T) : \mathfrak{T} \rightarrow \mathfrak{SCEG}$  (called the *secondary left derived 2-functors*) by

$$\mathbf{L}_n(T)(A) := \mathbf{H}_n(T(P_*))$$

where  $P_*$  is a projective resolution of  $A$ . If one takes the Takeuchi-Ulbrich homology instead and get the Takeuchi-Ulbrich left derived functors, which are denote by  $L_n T$ ,  $n \geq 0$ . So by the very definition one has  $\pi_0(\mathbf{L}_n(T)) = L_n T$ . It follows then that  $\pi_1(\mathbf{L}_n(T)) = L_{n+1} T$  (see Remark 3.2 in [13]).

In the following proposition we summarize the basic properties of derived functors.

**Proposition 6.** *i) The secondary derived functors form a system of  $\partial$ -functors. More precisely, if*

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \uparrow & \searrow & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

*is an extension in  $\mathfrak{T}$  (see Definition 2.2 in [7]), then the sequence of symmetric categorical groups*

$$\cdots \rightarrow \mathbf{L}_{n+1}T(C) \rightarrow \mathbf{L}_nT(A) \rightarrow \mathbf{L}_nT(B) \rightarrow \mathbf{L}_nT(C) \rightarrow \cdots$$

*is 2-exact. Furthermore we have the following exact sequence of abelian groups*

$$\cdots \rightarrow L_{n+1}T(C) \rightarrow L_nT(A) \rightarrow L_nT(B) \rightarrow L_nT(C) \rightarrow \cdots$$

*ii) If  $A$  is any object of  $\mathfrak{T}$ , then*

$$\mathbf{L}_nT(A) = \begin{cases} 0, & n < -1 \\ \Sigma \mathbf{L}_0T(A), & n = -1 \end{cases}$$

*Hence  $L_nT = 0$  if  $n < 0$ .*

*iii) If  $P$  is projective, then*

$$\mathbf{L}_nT(P) \cong \begin{cases} T(P), & n = 0 \\ \Omega T(P) & n = 1 \\ 0, & n \neq -1, 0, 1 \end{cases}$$

Furthermore

$$L_n T(P) \cong \begin{cases} \pi_i(TP) & n = 0, 1 \\ 0 & n \neq 0, 1 \end{cases}$$

iv) Assume  $T_n : \mathfrak{T} \rightarrow \mathfrak{SCE}$ ,  $n \in \mathbb{Z}$  is a system of  $\partial$ -functors such that  $T_n = 0$ , if  $n < -1$ . If for any projective  $P$  one has  $T_n(P) = 0$  for  $n > 1$  and  $\pi_1 T_1(P) = 0$ , then there exist a natural equivalence of 2-functors

$$\mathbf{L}_n T_0 \cong T_n, \quad n \in \mathbb{Z}$$

*Proof.* i) Easily follows from the fact that if  $f_* : P_* \rightarrow P'_*$  is a morphism of projective resolutions, which induces  $f$  in the zeroth homology, then the mapping cone of  $f$  is a projective resolution of  $C$ . ii) The fact  $\mathbf{L}_n = 0$  for  $n \leq -2$  is obvious. Applying Lemma 4 to the 2-exact sequence from i) we first get  $\pi_0(\mathbf{L}_{-1}(C)) = \pi_1(\mathbf{L}_{-2}(C)) = 0$  and then  $\pi_0(\mathbf{L}_0(C)) = \pi_1(\mathbf{L}_{-1}(C))$ . The statement iii) is clear. To show iv) observe that the argument in ii) shows that  $T_{-1}(A) \cong \Sigma T_0(A)$ . Next, we choose an extension

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \uparrow \alpha & \searrow & \\ B & \xrightarrow{f} & P & \xrightarrow{g} & A \end{array}$$

with projective  $P$ . If one applies the long exact sequence we get equivalence

$$T_{n+1}A \cong T_n B, \quad n \geq 2$$

and the following 2-exact sequence

$$0 \rightarrow T_2(A) \rightarrow T_1(B) \rightarrow T_1(P) \rightarrow T_1(A) \rightarrow T_0(B) \xrightarrow{f_*} T_0(P)$$

We claim that the canonical morphism  $T_1 A \rightarrow \ker(f_*)$  is an equivalence. By 2-exactness it suffice to show that the induced map  $\pi_1 T_1(A) \rightarrow \pi_1 T_0(B)$  is a monomorphism, but this follows from the fact that  $T_1 P$  is discrete. Hence all functors  $T_n$  can be express in terms of  $T_0$  and we are done.  $\square$

Observe that theory of left derived functors have obvious analogue for right derived 2-functors, both for covariant and contravariant case. Of course in covariant case we have to use injective resolutions instead of projective ones.

**Proposition 7.** i) The secondary right derived functors of a 2-functor  $T : \mathfrak{T} \rightarrow \mathfrak{SCE}$  form a system of  $\delta$ -functors. More precisely, if

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \uparrow \alpha & \searrow & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

is an extension in  $\mathfrak{T}$ , then the sequence of symmetric categorical groups

$$\cdots \rightarrow \mathbf{R}^n T(A) \rightarrow \mathbf{R}^n T(B) \rightarrow \mathbf{R}^n T(C) \rightarrow \mathbf{R}^{n+1} T(A) \rightarrow \cdots$$

is 2-exact. Moreover for any object  $A$  of  $\mathfrak{T}$ , one has

$$\mathbf{R}^n T(A) = \begin{cases} 0, & n < -1 \\ \Omega \mathbf{R}^0 T(A), & n = -1 \end{cases}$$

Furthermore, if  $I$  is injective, then

$$\mathbf{R}^n T(I) \cong \begin{cases} T(I), & n = 0 \\ \Sigma T(I) & n = 1 \\ 0, & n \neq -1, 0, 1 \end{cases}$$

ii) Assume  $T^n : \mathfrak{T} \rightarrow \mathfrak{SCG}$ ,  $n \in \mathbb{Z}$  is a system of  $\delta$ -functors such that  $T^n = 0$ , if  $n < -1$ . If for any injective  $P$  one has  $T^n(I) = 0$  for  $n > 1$  and  $\pi_0 T^1(I) = 0$ , then there exist a natural equivalence of 2-functors

$$\mathbf{R}^n T^0 \cong T^n, \quad n \in \mathbb{Z}$$

## 6. APPLICATIONS TO $\mathbf{Ext}$

In particular one can take the 2-functors  $\mathbf{Hom}(A, -)$  or  $\mathbf{Hom}(-, B)$  and get the secondary derived 2-functors. As in the classical case these two approach gives equivalent objects. Moreover we will show that in dimension 1 we recover the  $\mathbf{Ext}^1$  from [7].

To start with we let  $\mathbf{Ext}_{\mathfrak{T}}^n(A, -)$ ,  $n \in \mathbb{Z}$  be the secondary right derived functors of the 2-functors  $\mathbf{Hom}_{\mathfrak{T}}(A, -)$ . We use  $Ext_{\mathfrak{T}}^n(A, -)$  for Takeuchi-Ulbrich derived functors. Of course we are assuming that  $\mathfrak{T}$  has enough injective objects. Then by the dual of Proposition 6 we have:

**Proposition 8.** i) There are natural equivalences

$$\mathbf{Ext}_{\mathfrak{T}}^n(A, B) \cong \begin{cases} 0 & n \leq -2 \\ \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, \Omega B) = \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(\Sigma A, B) & n = -1 \\ \mathbf{Hom}_{\mathfrak{T}}(A, B), & n = 0 \end{cases}$$

In dimension  $-1$  it is understood that the abelian group  $\mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, \Omega B)$  is considered as a discrete symmetric categorical group. Thus

$$Ext_{\mathfrak{T}}^n(A, B) \cong \begin{cases} 0 & n \leq -2 \\ \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, \Omega B) = \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(\Sigma A, B) & n = -1 \\ \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, B), & n = 0 \end{cases}$$

ii) If  $I$  is an injective object in  $\mathfrak{T}$ , then  $\mathbf{Ext}_{\mathfrak{T}}^n(A, I) = 0$  for  $n > 1$  and  $\mathbf{Ext}_{\mathfrak{T}}^1(A, I)$  is a connected symmetric categorical group, with

$$\pi_1(\mathbf{Ext}_{\mathfrak{T}}^1(A, I)) \cong \mathrm{hom}_{\mathbf{Ho}(\mathfrak{T})}(A, B)$$

Thus

$$\text{Ext}_{\mathfrak{T}}^1(A, I) = 0$$

iii) Let

$$\begin{array}{ccc} & 0 & \\ & \uparrow \alpha & \\ B & \xrightarrow{f} C & \xrightarrow{g} D \end{array}$$

be an extension in  $\mathfrak{T}$ . Then the sequence

$$\cdots \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(A, B) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(A, C) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(A, D) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^{n+1}(A, B) \rightarrow \cdots$$

is a 2-exact sequence of symmetric categorical groups. Moreover the sequence

$$\cdots \rightarrow \text{Ext}_{\mathfrak{T}}^n(A, B) \rightarrow \text{Ext}_{\mathfrak{T}}^n(A, C) \rightarrow \text{Ext}_{\mathfrak{T}}^n(A, D) \rightarrow \text{Ext}_{\mathfrak{T}}^{n+1}(A, B) \rightarrow \cdots$$

is an exact sequence of abelian groups.

Assuming now  $\mathfrak{T}$  has enough projective objects as well. Then we have

**Proposition 9.** i) If  $P$  is an projective object in  $\mathfrak{T}$ , then  $\mathbf{Ext}_{\mathfrak{T}}^n(P, B) = 0$  for  $n > 1$  and  $\mathbf{Ext}_{\mathfrak{T}}^1(P, B)$  is a connected symmetric categorical group, with

$$\pi_1(\mathbf{Ext}_{\mathfrak{T}}^1(P, B)) \cong \text{hom}_{\mathbf{Ho}(\mathfrak{T})}(P, B)$$

Thus  $\text{Ext}_{\mathfrak{T}}^n(P, B) = 0$  for  $n > 0$ .

ii) Let

$$\begin{array}{ccc} & 0 & \\ & \uparrow \alpha & \\ A & \xrightarrow{f} B & \xrightarrow{g} C \end{array}$$

be an extension in  $\mathfrak{T}$ . Then the sequence

$$\cdots \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(C, D) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(B, D) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^n(A, D) \rightarrow \mathbf{Ext}_{\mathfrak{T}}^{n+1}(C, D) \rightarrow \cdots$$

is a 2-exact of symmetric categorical groups. Moreover the sequence

$$\cdots \rightarrow \text{Ext}_{\mathfrak{T}}^n(C, D) \rightarrow \text{Ext}_{\mathfrak{T}}^n(B, D) \rightarrow \text{Ext}_{\mathfrak{T}}^n(A, D) \rightarrow \text{Ext}_{\mathfrak{T}}^{n+1}(C, D) \rightarrow \cdots$$

is an exact sequence of abelian groups.

iii) The right derived functors of the 2-functor  $\mathbf{Hom}_{\mathfrak{T}}(-, B)$  are isomorphic to  $\mathbf{Ext}_{\mathfrak{T}}^n(-, B)$ .

iv) The categorical group  $\mathbf{Ext}_{\mathfrak{T}}^1(A, B)$  is isomorphic to the symmetric categorical group of extensions as it is defined in [7].

*Proof.* i) Since  $P$  is projective, the 2-functor  $\mathbf{Hom}_{\mathfrak{T}}(A, -)$  respects relative exact sequences and the result follows. ii) Take an injective resolution  $I^*$  of  $D$ , then  $\mathbf{Hom}_{\mathfrak{T}}(C, I^*) \rightarrow \mathbf{Hom}_{\mathfrak{T}}(B, I^*) \rightarrow \mathbf{Hom}_{\mathfrak{T}}(A, -)$  is an extension of 2-chain complexes and the result follows from [13]. iii) By i) and ii) the result follows from iv) of Proposition 6 and finally iv) follows from Corollary 11.3 in [7] and the formula for first derived functor obtained in the proof of iv) 6.

□

## 7. THE HOMOTOPY CATEGORY OF $\mathfrak{SCG}$

For symmetric categorical groups  $\mathbb{S}_1$  and  $\mathbb{S}_2$  we have a groupoid (in fact a symmetric categorical group [7])  $\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)$ . It follows from the result of [31] that the 2-category  $\mathfrak{SCG}$  is 2-equivalent to the 2-category of two-stage spectra (see also Proposition B.12 in [19]). Hence we can use the classical facts of algebraic topology to study  $\mathfrak{SCG}$ . Let  $\mathbf{\Gamma AB}$  be the category of triples  $(A, B, a)$  where  $A$  and  $B$  are abelian groups and

$$a \in \text{hom}(A/2A, B) = \text{hom}(A, {}_2B)$$

where  ${}_2B = \{b \in B \mid 2b = 0\}$ . A morphism  $(A, B, a) \rightarrow (A_1, B_1, a_1)$  is a pair  $(f, g)$  where  $f : A \rightarrow A_1$  and  $g : B \rightarrow B_1$  are homomorphisms, such that  $a_1 f = ga$ . The functor

$$k : \mathbf{Ho}(\mathfrak{SCG}) \rightarrow \mathbf{\Gamma AB}$$

is defined by

$$k(\mathbb{S}) := (\pi_0(\mathbb{S}), \pi_1(\mathbb{S}), k_{\mathbb{S}})$$

where  $\mathbb{S}$  is a symmetric categorical group and  $k_{\mathbb{S}}$  is the homomorphism induced by the commutativity constraints in  $\mathbb{S}$ .

**Proposition 10.** *For any symmetric categorical groups  $\mathbb{S}_1$  and  $\mathbb{S}_2$  one has a short exact sequence of abelian groups*

$$(1) \quad 0 \rightarrow \text{Ext}(\pi_0(\mathbb{S}_1), \pi_1(\mathbb{S}_2)) \rightarrow \pi_0(\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)) \rightarrow \mathbf{\Gamma AB}(k(\mathbb{S}_1), k(\mathbb{S}_2)) \rightarrow 0$$

*Furthermore one has also an isomorphism of abelian groups*

$$(2) \quad \pi_1(\mathbf{Hom}(\mathbb{S}_1, \mathbb{S}_2)) \cong \text{hom}(\pi_0(\mathbb{S}_1), \pi_1(\mathbb{S}_1))$$

*Proof.* The second isomorphism is obvious, while the first one is Proposition 7.1.6 in [5]. □

We see that the both categories  $\mathbf{Ho}$  and  $\mathbf{\Gamma AB}$  are additive and the functor

$$k : \mathbf{Ho} \rightarrow \mathbf{\Gamma AB}$$

is additive. In fact it is a part of a linear extension of categories (see Lemma 7.2.4 and Theorem 7.2.7 in [5]). It follows from the properties of linear extensions of categories [4] that the functor  $k$  is full, reflects isomorphisms, is essentially surjective on objects and it induces a bijection on the isomorphism classes of objects. Moreover the kernel of  $k$  (morphisms which goes to zero) is a square zero ideal of  $\mathbf{Ho}$ . Hence, for a given object  $\mathbb{A}$  of the category  $\mathbf{\Gamma AB}$  we can choose a symmetric categorical group  $K(\mathbb{A})$  such that  $k(K(\mathbb{A})) = \mathbb{A}$ . Such object exist and is unique up to equivalence. Moreover, for any morphism  $f : \mathbb{A} \rightarrow \mathbb{B}$  we can choose a morphism of symmetric categorical groups  $K(f) : K(\mathbb{A}) \rightarrow K(\mathbb{B})$ , such that  $k(K(f)) = f$ . The reader must be aware that the assignments  $\mathbb{A} \rightarrow H(\mathbb{A}), f \mapsto K(f)$  does NOT define a functor  $\mathbf{\Gamma AB} \rightarrow \mathbf{Ho}$ . Having in mind relation with spectra, the construction  $K$  for the objects of the form  $(A, 0, 0)$  coincides with Eilenberg-MacLane spectrum and in general case is consistent with Definition 7.1.5 in [5].

8. PROJECTIVE OBJECTS IN  $\mathfrak{SCG}$ 

In this section we prove the following theorem. Let us recall that the symmetric categorical group  $\Phi$  was defined in the introduction.

**Theorem 11.** *i) The symmetric categorical group  $\Phi$  is a small projective generator in  $\mathfrak{SCG}$ . In particular  $\mathfrak{SCG}$  has enough projective objects. Moreover, any projective object is equivalent to a coproduct of copies of  $\Phi$ .*

*ii) For any 2-ring  $R$  the right module  $R$  is a small projective generator of the abelian 2-category of 2-modules, in particular the abelian 2-category of 2-modules has enough projective objects.*

The statement on 2-modules is a direct consequence of Yoneda lemma for 2-categories. The statement on  $\mathfrak{SCG}$  is a consequence of Lemma 12 proved below.

Thanks to [21] a morphism  $f$  in  $\mathfrak{SCG}$  is faithful (resp. cofaithful) if underlying functor is faithful (resp. essentially surjective). Recall also that [21] a morphism  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  in  $\mathfrak{SCG}$  is essentially surjective if it is epimorphism on  $\pi_0$ , while a morphism  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  in  $\mathfrak{SCG}$  is faithful if it is monomorphism on  $\pi_1$ . We can develop same sort of language in the category  $\mathbf{\Gamma AB}$ . A morphism  $f = (f_0, f_1)$  in  $\mathbf{\Gamma AB}$  is *essentially surjective* if  $f_0$  is epimorphism of abelian groups. Moreover an object  $\mathbb{P}$  in  $\mathbf{\Gamma AB}$  is *projective* if for any essentially surjective morphism  $f : \mathbb{A} \rightarrow \mathbb{B}$  in  $\mathbf{\Gamma AB}$  the induced map

$$\mathbf{\Gamma AB}(\mathbb{P}, \mathbb{A}) \rightarrow \mathbf{\Gamma AB}(\mathbb{P}, \mathbb{B})$$

is surjective.

It is clear that a morphism  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  of symmetric categorical groups is *essentially surjective* iff  $k(F) : k(\mathbb{S}_1) \rightarrow k(\mathbb{S}_2)$  is so in  $\mathbf{\Gamma AB}$ . For an abelian group  $M$  we introduce two objects in  $\mathbf{\Gamma AB}$ :

$$l(M) := (M, M/2M, id_{M/2M}),$$

$$M[0] = (M, 0, 0).$$

**Lemma 12.** *i) If  $M$  is an abelian group and  $\mathbb{A} = (A_0, A_1, \alpha)$  is an object in  $\mathbf{\Gamma AB}$ , then one has the following functorial isomorphism of abelian groups*

$$\mathbf{\Gamma AB}(l(M), \mathbb{A}) = \text{hom}(M, A_0).$$

*ii) An object  $\mathbb{P}$  is projective in  $\mathbf{\Gamma AB}$  iff it is isomorphic to the object of the form  $l(P)$  with free abelian group  $P$ .*

*iii)  $\Phi$  is a projective object in  $\mathfrak{SCG}$  and any projective object in  $\mathfrak{SCG}$  is equivalent to a coproduct of  $\Phi$ .*

*iv) The 2-category of symmetric categorical groups have enough projective objects.*

*Proof.* i) Assume  $f = (f_0, f_1) : l(M) \rightarrow \mathbb{A}$  is a morphism in  $\mathbf{\Gamma AB}$ . So  $f_0 : M \rightarrow A_0$  and  $f_1 : M/2M \rightarrow A_1$  are homomorphisms of abelian groups and the following diagram is

commute

$$\begin{array}{ccc} M/2M & \xrightarrow{\text{Id}} & M/2M \\ \downarrow \hat{f}_0 & & \downarrow f_1 \\ A_0/2A_0 & \xrightarrow{\alpha} & A_1 \end{array}$$

Here  $\hat{f}_0$  is induced by  $f_0$ . It follows that  $f_1$  is completely determined by  $f_0$ . This proves the result.

ii) Let  $P$  be a free abelian group and let  $\mathbb{A} \rightarrow \mathbb{B}$  be an essentially surjective morphism in  $\mathbf{\Gamma AB}$ . Thus  $A_0 \rightarrow B_0$  is an epimorphism. It follows that  $\text{hom}(P, A_0) \rightarrow \text{hom}(P, B_0)$  is epimorphism as well, hence by the virtue of i) the map  $\mathbf{\Gamma AB}(l(P), \mathbb{A}) \rightarrow \mathbf{\Gamma AB}(l(P), \mathbb{B})$  is surjective. Thus  $l(P)$  is projective in  $\mathbf{\Gamma AB}$ . Conversely, assume  $\mathbb{P} = (P_0, P_1, \pi)$  is a projective object in  $\mathbf{\Gamma AB}$ . We claim that  $P_0$  is a free abelian group. In fact it suffice to show that it is a projective object in the category  $\mathbf{Ab}$  of abelian groups. Take any epimorphism of abelian groups  $f_0 : A \rightarrow B$  and any homomorphism of abelian groups  $g_0 : P_0 \rightarrow B$ . We have to show that  $g_0$  has a lift to  $A$ . Observe that  $f = (f_0, 0) : A[0] \rightarrow B[0]$  is essentially surjective in  $\mathbf{\Gamma AB}$  and  $g = (g_0, 0) : \mathbb{P} \rightarrow B[0]$  is a well-defined morphism in  $\mathbf{\Gamma AB}$ . By assumption we can lift  $g$  to a morphism  $\tilde{g} : \mathbb{P} \rightarrow A[0]$ . It is clear that  $\tilde{g} = (\tilde{g}_0, 0)$  for some  $\tilde{g}_0 : P_0 \rightarrow A$ . Clearly  $g_0 = f_0 \circ \tilde{g}_0$ . It follows that  $P_0$  is a free abelian group. Hence  $l(P_0)$  is a projective object in  $\mathbf{\Gamma AB}$ . By i) the identity map defines a canonical morphism  $i = (\text{Id}_{P_0}, i_1) : l(P_0) \rightarrow \mathbb{P}$ , which obviously is essentially surjective in  $\mathbf{\Gamma AB}$ . Since  $\mathbb{P}$  is projective it follows that there exist a morphism  $p = (\text{Id}_{P_0}, p_1) : \mathbb{P} \rightarrow l(P)$  such that  $i \circ p = \text{Id}_{\mathbb{P}}$ . Thus we have a commutative diagram

$$\begin{array}{ccc} P_0/2P_0 & \xrightarrow{\pi} & P_1 \\ \downarrow \text{Id} & & \downarrow p_1 \\ P_0/2P_0 & \xrightarrow{\text{Id}} & P_0/2P_0 \\ \downarrow \text{Id} & & \downarrow i_1 \\ P_0/2P_0 & \xrightarrow{\pi} & P_1 \end{array}$$

with  $i_1 p_1 = \text{Id}_{P_1}$ . It follows that  $p_1$  and  $i_1$  are mutually inverse isomorphisms of abelian groups. Hence  $p : \mathbb{P} \rightarrow l(P)$  and  $l : l(P) \rightarrow \mathbb{P}$  are mutually inverse isomorphisms in  $\mathbf{\Gamma AB}$ .

iii) First of all observe that  $\mathbf{k}(\Phi) = l(\mathbb{Z})$ . Hence our assertion is equivalent to the following one: For any free abelian group  $P$  the symmetric categorical group  $K(l(P))$  is projective symmetric categorical group and conversely, if  $\mathbb{S}$  is a projective symmetric categorical group then  $\pi_0(\mathbb{S})$  is a free abelian group  $\mathbb{S}$  is equivalent to  $H(l(\pi_0(\mathbb{S})))$ . To prove it, let  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  be an essentially surjective morphism of symmetric categorical groups and  $G : K(l(P)) \rightarrow \mathbb{S}_2$  be a morphism of symmetric categorical groups. Apply the functor  $\mathbf{k}$  to get morphisms  $\mathbf{k}(F) : \mathbf{k}(\mathbb{S}_1) \rightarrow \mathbf{k}(\mathbb{S}_2)$  and  $\mathbf{k}(G) : l(P) \rightarrow \mathbf{k}(\mathbb{S}_2)$  in  $\mathbf{\Gamma AB}$ . Since  $\pi_0(F) : \pi_0(\mathbb{S}_1) \rightarrow \pi_0(\mathbb{S}_2)$  is an epimorphism of abelian groups it follows that  $\mathbf{k}(F) : \mathbf{k}(\mathbb{S}_1) \rightarrow \mathbf{k}(\mathbb{S}_2)$  is an essentially surjective morphism in  $\mathbf{\Gamma AB}$ . Since  $P$  is a free abelian group  $l(P)$  is



projective in  $\mathbf{\Gamma AB}$  by ii). Thus we can lift  $k(F)$  to get a morphism  $\hat{g} : l(P) \rightarrow k(\mathbb{S}_1)$  such that  $k(F) \circ \hat{g} = k(G)$  holds in  $\mathbf{\Gamma AB}(l(P), k(\mathbb{S}_2))$ . Since  $P = \pi_0(K(l(P)))$  is free abelian group the Ext-term in the exact sequence (1) disappears and we get the isomorphism

$$(3) \quad \pi_0(\mathbf{Hom}(K(l(P)), \mathbb{S}_i)) \cong \mathbf{\Gamma AB}(l(P), k(\mathbb{S}_i)), \quad i = 0, 1$$

Take a morphism  $L : K(l(P)) \rightarrow \mathbb{S}_1$  of symmetric categorical groups which corresponds to the morphism  $\hat{g} : l(P) \rightarrow k(\mathbb{S}_1)$ . By our construction one has an equality  $k(FL) = k(G)$  in  $\mathbf{\Gamma AB}(l(P), k(\mathbb{S}_2)) = \pi_0(\mathbf{Hom}(H(l(P)), \mathbb{S}_2))$ . Thus the classes of  $FL$  and of  $G$  in  $\pi_0(\mathbf{Hom}(K(l(P)), \mathbb{S}_1))$  are the same. Hence there exist a track from  $FL$  to  $G$ . This shows that  $K(l(P))$  is a projective symmetric categorical group. Conversely assume  $\mathbb{S}$  is a projective symmetric categorical group. Since  $\mathbb{S}$  and  $K(k(\mathbb{S}))$  are equivalent, it follows that  $K(k(\mathbb{S}))$  is also projective. We claim that  $k(\mathbb{S})$  is projective in  $\mathbf{\Gamma AB}$ . In fact take any essentially surjective morphism  $f = (f_0, f_1) : \mathbb{A} \rightarrow \mathbb{B}$  and any morphism  $g : k(\mathbb{S}) \rightarrow \mathbb{B}$  in  $\mathbf{\Gamma AB}$ . Then  $K(f) : K(\mathbb{A}) \rightarrow K(\mathbb{B})$  is essentially surjective in  $\mathbf{SCG}$ . Hence for  $K(g) : K(k(\mathbb{S})) \rightarrow K(\mathbb{B})$  we have a morphism  $\tilde{G} : K(k(\mathbb{S})) \rightarrow K(\mathbb{A})$  and a track  $K(f) \circ \tilde{G} \rightarrow K(g)$ . Thus  $K(f) \circ \tilde{G} = K(g)$  in  $\pi_0(\mathbf{Hom}(K(k(\mathbb{S})), K(\mathbb{B})))$ . Now apply the functor  $k$  to get the equality  $f \circ k(\tilde{G}) = g$ , showing that  $k(\mathbb{S})$  is projective in  $\mathbf{\Gamma AB}$ . Hence  $k(\mathbb{S})$  is isomorphic to  $l(P)$  for a free abelian group  $P$ . Thus  $\mathbb{S}$  and  $K(l(P))$  are equivalent.

iv) Let  $\mathbb{S}$  be a symmetric categorical group. Choose a free abelian group  $P$  and an epimorphism of abelian groups  $f_0 : P \rightarrow \pi_0(\mathbb{S})$ . By Lemma 12 it has a unique extension to a morphism  $f = (f_0, f_1) : l(P) \rightarrow k(\mathbb{S})$  which is essentially surjective. Since  $P$  is a free abelian group, we have the isomorphism (3), which show that there exist a morphism of symmetric categorical groups  $K(l(P)) \rightarrow \mathbb{S}$  which realizes  $f_0$  on the level of  $\pi_0$ . Clearly this morphism does the job.  $\square$

**Proposition 13.** *The 2-category of symmetric categorical groups is 2-equivalent to the category of right categorical modules over the categorical ring  $\Phi$ .*

*Proof.* Since  $\Phi$  is a small projective generator the 2-category of symmetric categorical groups is 2-equivalent to the category of right categorical modules over the categorical ring  $\mathbf{Hom}(\Phi, \Phi)$ . Observe that we have an obvious morphisms of 2-rings  $\Phi \rightarrow \mathbf{Hom}(\Phi, \Phi)$  which sends the object  $n$  to the endomorphism of  $\Phi$  which on objects is given by  $x \mapsto nx$ . It remains to show that this morphism of 2-rings is an equivalence. But this easily follows from the exact sequence 1.  $\square$

## 9. INJECTIVE OBJECTS

In this section we prove the following result.

**Theorem 14.** *The abelian category  $\mathbf{SCG}$  as well as the abelian 2-category of 2-modules over a 2-ring have enough injective objects.*

These are just part iv) and v) of Lemma 15 proved below.

A morphism  $f = (f_0, f_1)$  in  $\mathbf{\Gamma AB}$  is *faithful* provided  $f_1$  is injective and an object  $\mathbb{I} = (I_0, I_1, \iota)$  of  $\mathbf{\Gamma AB}$  is *injective* if for any faithful morphism  $f : \mathbb{A} \rightarrow \mathbb{B}$  in  $\mathbf{\Gamma AB}$  the

induced map

$$\mathbf{\Gamma AB}(\mathbb{B}, \mathbb{I}) \rightarrow \mathbf{\Gamma AB}(\mathbb{A}, \mathbb{I})$$

is surjective. It is clear that a morphism  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  of symmetric categorical groups is faithful iff  $k(F) : k(\mathbb{S}_1) \rightarrow k(\mathbb{S}_2)$  is faithful in  $\mathbf{\Gamma AB}$ . For an abelian group  $M$  we introduce two objects in  $\mathbf{\Gamma AB}$ :

$$\begin{aligned} r(M) &= ({}_2M, M, id_{{}_2M}), \\ M[1] &= (0, M, 0). \end{aligned}$$

**Lemma 15.** *i) If  $M$  is an abelian group and  $\mathbb{A} = (A_0, A_1, \alpha)$  is an object in  $\mathbf{\Gamma AB}$ , then one has the following functorial isomorphism of abelian groups*

$$\mathbf{\Gamma AB}(\mathbb{A}, r(M)) = \text{hom}(A_1, M).$$

*ii) An object  $\mathbb{Q}$  is an injective object in  $\mathbf{\Gamma AB}$  iff it is isomorphic to the object of the form  $r(Q)$  with divisible abelian group  $Q$ .*

*iii) For any divisible abelian group  $Q$  the symmetric categorical group  $K(r(Q))$  is injective. Conversely, if  $\mathbb{S}$  is an injective categorical group then  $\pi_1(\mathbb{S})$  is a divisible abelian group and  $\mathbb{S}$  is equivalent to  $K(r(\pi_1(\mathbb{S})))$ .*

*iv) The 2-category of symmetric categorical groups have enough injective objects.*

*v) Let  $\mathbb{R}$  be a categorical group. Then the category of categorical right  $\mathbb{R}$ -modules have enough injective objects.*

*Proof.* i) Assume  $g = (g_0, g_1) : \mathbb{A} \rightarrow r(M)$  is a morphism in  $\mathbf{\Gamma AB}$  i in  $\mathbf{\Gamma AB}$ . So  $g_0 : A_0 \rightarrow {}_2M$  and  $g_1 : A_1 \rightarrow M$  are homomorphisms of abelian groups and we have a commutative diagram:

$$\begin{array}{ccccc} A_0 & \xrightarrow{\alpha} & {}_2A_1 & \xrightarrow{i} & A_1 \\ g_0 \downarrow & & \bar{g}_1 \downarrow & & \downarrow g_1 \\ {}_2M & \xrightarrow{\text{Id}} & {}_2M & \xrightarrow{j} & M \end{array}$$

where  $\bar{g}_1$  is induced by  $g_1$  and  $i, j$  are inclusions. It follows that  $g_0$  is completely determined by  $g_1$  and the result follows.

ii) Let  $\mathbb{Q}$  be a divisible abelian group and let  $\mathbb{A} \rightarrow \mathbb{B}$  be a faithful morphism in  $\mathbf{\Gamma AB}$ . Thus  $A_1 \rightarrow B_1$  is a monomorphism. Since  $\mathbb{Q}$  is an injective object in  $\mathbf{\mathfrak{Ab}}$  it follows that  $\text{hom}(B_1, \mathbb{Q}) \rightarrow \text{hom}(A_1, \mathbb{Q})$  is an epimorphism of abelian groups. So by i) the map  $\mathbf{\Gamma AB}(\mathbb{B}, r(\mathbb{Q})) \rightarrow \mathbf{\Gamma AB}(\mathbb{A}, r(\mathbb{Q}))$  is surjective. Thus  $r(\mathbb{Q})$  is injective in  $\mathbf{\Gamma AB}$ .

Conversely, assume  $\mathbb{Q} = (Q_0, Q_1, \chi)$  is an injective in  $\mathbf{\Gamma AB}$ . We claim that  $Q_1$  is a divisible abelian group. In fact it suffice to show that it is an injective object in the category  $\mathbf{\mathfrak{Ab}}$ . Take any monomorphism of abelian groups  $f_1 : A \rightarrow B$  and any homomorphism of abelian groups  $g_1 : A_1 \rightarrow Q_0$ . We have to show that  $g_1$  has a lift to  $B_1$ . Observe that  $f = (0, f_1) : A[1] \rightarrow B[1]$  is faithful in  $\mathbf{\Gamma AB}$  and  $g = (0, g_1) : A[1] \rightarrow \mathbb{Q}$  is a well-defined morphism in  $\mathbf{\Gamma AB}$ . By assumption there exists a morphism  $\tilde{g} : B[1] \rightarrow \mathbb{Q}$ . It is clear that  $\tilde{g} = (0, \tilde{g}_1)$  for some  $\tilde{g}_1 : B \rightarrow Q_1$ . Thus  $Q_1$  is a divisible abelian group. Thus  $r(Q_1)$  is an injective object in  $\mathbf{\Gamma AB}$ . By i) the identity map defines a canonical morphism  $i = (i_0, \text{Id}_{Q_1}) : \mathbb{Q} \rightarrow r(Q_1)$ , which obviously is faithful in  $\mathbf{\Gamma AB}$ . Since  $\mathbb{Q}$  is injective it

follows that there exist a morphism  $q = (q_0, \text{Id}_{Q_1}) : r(Q_1) \rightarrow \mathbb{Q}$  such that  $q \circ i = \text{Id}_{\mathbb{Q}}$ . Thus we have a commutative diagram

$$\begin{array}{ccc} Q_0 & \xrightarrow{\chi} & {}_2Q_1 \\ \downarrow i_0 & & \downarrow \text{Id} \\ {}_2Q_1 & \xrightarrow{\text{Id}} & {}_2Q_1 \\ \downarrow q_0 & & \downarrow \text{Id} \\ Q_0 & \xrightarrow{\chi} & {}_2Q_1 \end{array}$$

with  $q_0 i_0 = \text{Id}_{Q_0}$ . It follows that  $i_0$  is an isomorphism. Hence  $i : \mathbb{Q} \rightarrow r(Q_1)$  is an isomorphism and we are done.

iii) Let  $F : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  be a faithful morphism of symmetric categorical groups and  $G : \mathbb{S}_1 \rightarrow K(r(Q))$  be a morphism of symmetric categorical groups. Apply the functor  $\mathbf{k}$  to get morphisms  $\mathbf{k}(F) : \mathbf{k}(\mathbb{S}_1) \rightarrow \mathbf{k}(\mathbb{S}_2)$  and  $\mathbf{k}(G) : \mathbf{k}(\mathbb{S}_1) \rightarrow r(Q)$  in  $\mathbf{\Gamma AB}$ . Since  $\pi_1(F) : \pi_1(\mathbb{S}_1) \rightarrow \pi_1(\mathbb{S}_2)$  is a monomorphism of abelian groups it follows that  $\mathbf{k}(F) : \mathbf{k}(\mathbb{S}_1) \rightarrow \mathbf{k}(\mathbb{S}_2)$  is a faithful morphism in  $\mathbf{\Gamma AB}$ . Since  $Q$  is a divisible abelian group,  $r(Q)$  is injective in  $\mathbf{\Gamma AB}$  by ii) and we can extend  $\mathbf{k}(F)$  to get a morphism  $\hat{g} : \mathbf{k}(\mathbb{S}_2) \rightarrow r(Q)$ . Thus we have the equality  $\hat{g} \circ \mathbf{k}(F) = \mathbf{k}(G)$  in  $\mathbf{\Gamma AB}(\mathbf{k}(\mathbb{S}_1), r(Q))$ . Since  $Q = \pi_1(K(r(Q)))$  is divisible abelian group the Ext-term in the exact sequence (1) disappears and we get the isomorphism

$$(4) \quad \pi_0(\mathbf{Hom}(\mathbb{S}_i, K(r(Q)))) \cong \mathbf{\Gamma AB}(\mathbf{k}(\mathbb{S}_1), r(Q)), \quad i = 0, 1$$

Take a morphism  $L : \mathbb{S}_2 \rightarrow K(r(Q))$  of symmetric categorical groups which corresponds to the morphism  $\hat{g} : \mathbf{k}(\mathbb{S}_2) \rightarrow r(Q)$ . By our construction one has an equality  $\mathbf{k}(LF) = \mathbf{k}(G)$ . This is equality in  $\pi_0(\mathbf{Hom}(\mathbb{S}_1, K(r(Q))))$ , which imply that the classes of  $LF$  and of  $G$  in  $\pi_0(\mathbf{Hom}(\mathbb{S}_1, K(r(Q))))$  are the same. Thus there exist a track from  $LF$  to  $G$ . This shows that  $K(r(Q))$  is an injective symmetric categorical group. Conversely assume  $\mathbb{S}$  is an injective symmetric categorical group. Since  $\mathbb{S}$  and  $K(\mathbf{k}(\mathbb{S}))$  are equivalent, it follows that  $K(\mathbf{k}(\mathbb{S}))$  is also projective. We claim that  $\mathbf{k}(\mathbb{S})$  is injective in  $\mathbf{\Gamma AB}$ . In fact take any faithful morphism  $f = (f_0, f_1) : \mathbb{A} \rightarrow \mathbb{B}$  in  $\mathbf{\Gamma AB}$  and any morphism  $g : \mathbb{A} \rightarrow \mathbf{k}(\mathbb{S})$  in  $\mathbf{\Gamma AB}$ . Then  $K(f) : K(\mathbb{A}) \rightarrow K(\mathbb{B})$  is faithful in  $\mathbf{\mathfrak{SCG}}$ . Hence for  $K(g) : K(\mathbb{A}) \rightarrow K(\mathbf{k}(\mathbb{S}))$  we have a morphism  $\tilde{G} : K(\mathbb{B}) \rightarrow K(\mathbf{k}(\mathbb{S}))$  and a track  $\tilde{G} \circ K(f) \rightarrow K(g)$ . Thus  $\tilde{G} \circ K(f) = K(g)$  in  $\pi_0(\mathbf{Hom}(K(\mathbf{k}(\mathbb{S})), K(\mathbb{B})))$ . Now apply the functor  $\mathbf{k}$  to get the equality  $\mathbf{k}(\tilde{G}) \circ f = g$ , showing that  $\mathbf{k}(\mathbb{S})$  is injective in  $\mathbf{\Gamma AB}$ . Hence  $\mathbf{k}(\mathbb{S})$  is isomorphic to  $r(Q)$  for a divisible abelian group  $Q$ . Thus  $\mathbb{S}$  and  $K(r(Q))$  are equivalent.

iv) Let  $\mathbb{S}$  be a symmetric categorical group. Choose a divisible abelian group  $Q$  and monomorphism of abelian groups  $f_1 : \pi_1(\mathbb{S}) \rightarrow Q$ . By Lemma 15 it has a unique extension to a morphism  $f = (f_0, f_1) : \mathbf{k}(\mathbb{S}) \rightarrow r(Q)$  which is essentially surjective. Since  $Q$  is divisible abelian group, we have the isomorphism (4), which show that there exist a morphism of symmetric categorical groups  $\mathbb{S} \rightarrow K(r(Q))$  which realizes  $f_1$  on the level of  $\pi_1$  and we get the result.

v) We consider the 2-functor  $\mathbf{Hom}(\mathbb{R}, -)$  from the 2-category of symmetric categorical groups to the 2-category of categorical right  $R$ -modules. It is a right 2-adjoint to

the forgetful 2-functor. Since the forgetful functor is exact it follows that the 2-functor  $\mathbf{Hom}(\mathbb{R}, -)$  takes injective objects to injective ones. Let  $\mathbb{M}$  be a categorical left  $\mathbb{R}$ -module. Choose a faithful morphism  $\mathbb{M} \rightarrow \mathbb{Q}$  in the 2-category of symmetric categorical groups with injective symmetric categorical group  $\mathbb{Q}$ . Apply now the 2-functor  $\mathbf{Hom}(\mathbb{R}, -)$ . It follows from the isomorphism (2) that  $\mathbf{Hom}(\mathbb{R}, \mathbb{M}) \rightarrow \mathbf{Hom}(\mathbb{R}, \mathbb{Q})$  is a faithful morphism of right  $\mathbb{R}$ -modules. By the same reasons the obvious morphism  $\mathbb{M} \rightarrow \mathbf{Hom}(\mathbb{R}, \mathbb{M})$  is also faithful. Taking the composite we obtain a faithful morphism  $\mathbb{M} \rightarrow \mathbf{Hom}(\mathbb{R}, \mathbb{Q})$  and hence the result.  $\square$

## 10. ON COHOMOLOGY OF CATEGORICAL GROUPS

Let  $\mathbb{G}$  be a categorical group. Recall that a  $\mathbb{G}$ -module [15], [34] (or *2-representation of*  $\mathbb{G}$ ) is a symmetric categorical group  $\mathbb{S}$  together with a homomorphism of categorical groups  $\mathbb{G} \rightarrow \mathcal{E}q(\mathbb{S})$ , where  $\mathcal{E}q(\mathbb{S})$  is the categorical group of symmetric monoidal autoequivalences of  $\mathbb{S}$ . In the case, when  $\mathbb{G}$  is discrete, Ulbrich [34] defined the cohomology groups  $H_U^*(\mathbb{G}, \mathbb{S})$ . Moreover in [15],[16] the authors considered even more general situation when  $\mathbb{G}$  is arbitrary and  $\mathbb{S}$  is assumed to be only braided and they managed to define the categorical groups  $\mathcal{H}^i(\mathbb{G}, \mathbb{S})$  in dimensions  $i = 0$  and  $i = 1$ .

For symmetric categorical groups one can define  $\mathcal{H}^i(\mathbb{G}, \mathbb{S})$  for all  $i$ . For discrete  $\mathbb{G}$  the connected components of  $\mathcal{H}^i(\mathbb{G}, \mathbb{S})$  are exactly the Ulbrich's groups. The main result of this section claims that there is an equivalence between  $\mathcal{H}^*(\mathbb{G}, \mathbb{S})$  and appropriate ext in the abelian 2-category of  $\mathbb{G}$ -modules.

For any symmetric categorical group  $\mathbb{S}$ , we let  $M(\mathbb{G}, \mathbb{S})$  be the symmetric categorical group of all functors from  $\mathbb{G}$  to  $\mathbb{S}$ . In fact  $M(\mathbb{G}, \mathbb{S})$  has a natural  $\mathbb{G}$ -module structure, induced by the categorical group structure on  $\mathbb{G}$ . If  $\mathbb{S}$  is a  $\mathbb{G}$ -module, then there is a canonical morphism of  $\mathbb{G}$ -modules

$$i_{\mathbb{S}} : \mathbb{S} \rightarrow M(\mathbb{G}, \mathbb{S})$$

which takes an object  $a \in \mathbb{S}$  to the constant functor with value  $a$ . Since

$$\pi_1 M(\mathbb{G}, \mathbb{S}) = \text{Maps}(\pi_0(\mathbb{G}), \pi_1(\mathbb{S}))$$

we see that  $i_{\mathbb{S}}$  is faithful.

For a categorical group  $\mathbb{G}$  we let  $Ner_2(\mathbb{G})$  be the nerve of  $\mathbb{G}$  as it is defined in [9]. For a  $\mathbb{G}$ -module  $\mathbb{S}$  we let  $C^n(\mathbb{G}, \mathbb{S})$  be the symmetric categorical group  $\prod_{Ner_2(\mathbb{G})_n} \mathbb{S}$ . Similarly to the classical case, there is a 2-cochain complex structure on  $C^*(\mathbb{G}, \mathbb{S})$  (details can be find in [12]). The secondary cohomology of this complex is denoted by  $\mathcal{H}^*(\mathbb{G}, \mathbb{S})$ , while the Takeuchi-Ulbrich cohomology is denoted by  $H_U^*(\mathbb{G}, \mathbb{S})$ . By our definition the symmetric categorical groups  $\mathcal{H}^0(\mathbb{G}, \mathbb{S})$  and  $\mathcal{H}^1(\mathbb{G}, \mathbb{S})$  coincides with one defined in [15],[16], while for discrete  $\mathbb{G}$  the groups  $H_U^*(\mathbb{G}, \mathbb{S})$  are the same as in [34].

We wish to relate these objects to the secondary ext. To do so, we observe that the 2-category  $\mathfrak{SCB}_{\mathbb{G}}$  of  $\mathbb{G}$ -modules have enough injective objects. In fact if  $\mathbb{Q}$  is an injective object in  $\mathfrak{SCB}$ , considered as a trivial  $\mathbb{G}$ -module, then  $M(\mathbb{G}, \mathbb{Q})$  is injective in  $\mathfrak{SCB}_{\mathbb{G}}$ . Actually the 2-category  $\mathfrak{SCB}_{\mathbb{G}}$  of  $\mathbb{G}$ -modules has a small projective generator  $M_f(\mathbb{G}, \Phi)$ ,

and hence is 2-equivalent to the 2-category of modules over a 2-ring. We will not need this fact, therefore we omit the proof of this fact.

**Theorem 16.** *Let  $\mathbb{G}$  be a categorical group and  $\mathbb{S}$  be a  $\mathbb{G}$ -module, then*

$$\mathcal{H}^*(\mathbb{G}, \mathbb{S}) \cong \mathbf{Ext}_{\mathfrak{S}\mathfrak{C}\mathfrak{S}_{\mathbb{G}}}^*(\Phi, \mathbb{S})$$

where action of  $\mathbb{G}$  on  $\Phi$  is trivial.

*Proof.* By iv) Proposition 6 it suffices to prove the following assertions

- i)  $\mathbf{Hom}_{\mathfrak{S}\mathfrak{C}\mathfrak{S}_{\mathbb{G}}}(\Phi, -) \cong \mathcal{H}^0(\mathbb{G}, -)$ ,
- ii) If

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \alpha \uparrow & \searrow & \\ \mathbb{S}_1 & \xrightarrow{f} & \mathbb{S}_2 & \xrightarrow{g} & \mathbb{S}_3 \end{array}$$

is an extension in  $\mathfrak{S}\mathfrak{C}\mathfrak{S}_{\mathbb{G}}$ , then

$$\begin{array}{ccccc} & & 0 & & \\ & \nearrow & \alpha \uparrow & \searrow & \\ C^n(\mathbb{G}, \mathbb{S}_1) & \xrightarrow{f} & C^n(\mathbb{G}, \mathbb{S}_2) & \xrightarrow{g} & C^n(\mathbb{G}, \mathbb{S}_3) \end{array}$$

is also an extension.

- iii) If  $\mathbb{S}$  is an injective object in  $\mathfrak{S}\mathfrak{C}\mathfrak{S}_{\mathbb{G}}$ , then  $\mathcal{H}^n(\mathbb{G}, \mathbb{S}) = 0$  for  $n > 1$ .

The assertion i) is easy consequence of the fact that

$$\mathbf{Hom}_{\mathfrak{S}\mathfrak{C}\mathfrak{S}_{\mathbb{G}}}(\Phi, \mathbb{S}) \cong \mathbb{S}.$$

To see the ii) one has to use the fact that the product of 2-exact sequences is also 2-exact and to show iii) one has to consider the canonical morphism  $i_{\mathbb{S}} : S \rightarrow M(\mathbb{G}, \mathbb{S})$ . Since  $\mathbb{S}$  is injective and  $i_{\mathbb{S}}$  is faithful it follows from Corollary 4.4 [7] that  $S$  is equivalent to a direct summand of  $M(\mathbb{G}, \mathbb{S})$ . Hence it suffice to show that for any  $\mathbb{G}$ -module  $\mathbb{S}$  one has  $\mathcal{H}^n(\mathbb{G}, M(\mathbb{G}, \mathbb{S})) = 0$  for  $n > 1$ . Since 'evaluation at unite' gives an explicit homotopy equivalence between  $\mathbb{S}$  considered as a 2-chain complex concentrated in dimension zero and  $C^*(\mathbb{G}, \mathbb{S})$ , the result follows.  $\square$

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